

Renyi entropies of a black hole from Hawking radiation

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Abstract

Renyi entropies of a black hole are evaluated by counting the states of the Hawking radiation which fills a thin shell surrounding the horizon. The width of the shell is determined from its energy content and the corresponding mass defect. The Bekenstein-Hawking formula for the entropy of the black hole is correctly reproduced.

1 Introduction

The effective number of quantum states inside a black hole of mass M is determined by the Bekenstein-Hawking formula for its entropy [1, 2]

$$S_M = 4\pi M^2, \tag{1}$$

giving an important information about the probabilities p_i of the states ψ_i forming the black hole¹. Indeed, when p_i 's are introduced into the general

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¹Throughout this paper we use $G = \hbar = 1$.

formula for entropy of a statistical system

$$S = - \sum_i p_i \log p_i, \quad (2)$$

they have to satisfy (1).

To obtain more information about p_i it is necessary to investigate other quantities of similar nature. In the present paper we discuss the moments of p_i , i.e. the coincidence probabilities,

$$C_l = \sum_{i=1}^{\Gamma} [p_i]^l \quad (3)$$

where Γ is the total number of states of the system.

Since the coincidence probabilities are, generally, rather small numbers, it is more convenient to consider the Renyi entropies [3] defined as

$$H_l = - \frac{1}{1-l} \log C_l. \quad (4)$$

It is not difficult to show that

$$S = H_1 \quad (5)$$

where the R.H.S. is understood as the limiting value of H_l when $l \rightarrow 1$.

Our method of evaluation of H_l is based on the conjecture that the probability distribution p_i is encoded in the Hawking radiation emitted by the black hole. To be more precise, we evaluate the Renyi entropies of Hawking radiation emitted into a thin shell close to the horizon. The width of the shell is determined from the condition that the amount of the effective energy emitted into this shell is equal to the loss of the mass of the black hole.

It is shown that this procedure gives the correct Bekenstein-Hawking formula (1) for H_1 , i.e. the entropy of the black hole [*c.f.*(5)] and thus it is justified *a posteriori*.

Since the straightforward counting of states of the Hawking radiation leads to a singularity at the black hole horizon [4], one has to employ a certain regularization procedure. The singularity comes about because at the horizon ($r = \rho$, where $\rho = 2M$ is the black hole radius and r is the Schwarzschild coordinate) the g_{00} component of the metric tensor vanishes:

$$g_{00}(r; \rho) = 1 - \rho/r. \quad (6)$$

To deal with singularities we replace (6) in the region close to the horizon by

$$g_{00}(\rho \leq r \leq \rho(1 + \delta); \rho) = g_{00}(\rho(1 + \delta); \rho) \approx \delta \quad (7)$$

i.e. we take g_{00} to be a constant, starting from a certain arbitrarily small distance from the horizon. For $r \geq \rho(1 + \delta)$ (6) remains valid. It turns out that our final result is finite and does not depend on δ in the limit $\delta \rightarrow 0$. Thus our regularization procedure does not introduce any additional uncertainties.

To summarize, we evaluate the Renyi entropies of the Hawking radiation in a certain region of configuration space. This region is selected in such a way that its energy content reproduces the mass lost by the black hole through the radiation. It turns out that in this way the standard Bekenstein-Hawking formula for entropy is recovered. The same procedure is then employed for evaluation of other Renyi entropies.

In the next section the general formulae for entropies of the Bose gas are given. They are applied to black holes in Section 3. In Section 4 the appropriate volume is estimated and expressed in terms of mass defect. The final formulae for the Shannon and Renyi entropies are obtained in Section 5. Our conclusions are listed in the last section. Evaluation of some relevant integrals is described in the Appendix.

2 A general formula for Renyi entropies of a Bose gas

We follow closely the argument given in [5] where we have obtained the relation

$$H_l = \left(1 + \frac{1}{l} + \frac{1}{l^2} + \frac{1}{l^3}\right) \frac{S}{4}, \quad (8)$$

valid for the free photon gas closed in a large static volume (in the flat space). The main result of this paper is that the same formula is also valid for black holes.

The probability of having n_1 bosons in a state with energy ϵ_1 , n_2 bosons with energy ϵ_2, \dots is given by

$$P(n_1, n_2, \dots, n_M) = \prod_{m=1}^M \left(1 - e^{-\beta \epsilon_m}\right) e^{-\beta n_m \epsilon_m} \quad (9)$$

where $\beta = 1/T$ and we have put the chemical potential to zero.

The coincidence probabilities are

$$C_l = \sum_{n_1, n_2, \dots} [P(n_1, n_2, \dots, n_M)]^l = \prod_{m=1}^M \frac{(1 - e^{-\beta\epsilon_m})^l}{(1 - e^{-\beta l\epsilon_m})} \quad (10)$$

This gives for the Renyi entropies

$$H_l = \frac{1}{1-l} \log C_l = - \sum_{m=1}^M \log(1 - e^{-\beta\epsilon_m}) + \frac{1}{1-l} \sum_{m=1}^M \log\left(\frac{1 - e^{-\beta\epsilon_m}}{1 - e^{-\beta l\epsilon_m}}\right). \quad (11)$$

Finally, when the sum over states is replaced by an integral we have

$$H_l = \int dN(\epsilon) W_l(\beta\epsilon) \quad (12)$$

where $dN(\epsilon)$ is the number of states with energy between ϵ and $\epsilon + d\epsilon$ and

$$W_l(z) = -\log(1 - e^{-z}) + \frac{1}{1-l} \log\left(\frac{1 - e^{-z}}{1 - e^{-lz}}\right). \quad (13)$$

This formula was derived in [5].

3 Application to black holes

Our problem now is to evaluate H_l for the radiation emitted by the black hole of radius ρ into an infinitesimal layer around its horizon. To this end we first have to specify the physical meaning of the energy ϵ and of the temperature $T = 1/\beta$ in the case of a black hole. Since the black-body radiation is in equilibrium, the temperature T is a constant, provided ϵ is the energy *conserved* in the process²

This implies that T is the Hawking temperature

$$T = T_H = \frac{1}{4\pi\rho} = \frac{1}{8\pi M}. \quad (14)$$

To perform the integration in (13) we first have to determine the density of states $N(\epsilon)$. The relevant calculation was performed in [4] where the

²Since the gravitational field outside of the black hole is static, the conserved energy can be defined, c.f. [6], Section 88, and [7], Section 27.

formula for the number of states with energy smaller than ϵ was given in the form

$$N(\epsilon) = \frac{dr}{\pi g_{00}} \int_0^{l_{max}} (2l+1) dl \sqrt{\epsilon^2 - g_{00} \left(m^2 + \frac{l(l+1)}{r^2} \right)} \quad (15)$$

where g_{00} is given by (6). The integration extends for the values of l for which the argument of the square root is positive. This integral can be evaluated and one finally obtains

$$dN(\epsilon) = \frac{dN(\epsilon)}{d\epsilon} d\epsilon = \frac{2r^2 dr}{\pi [g_{00}]^2} \sqrt{\epsilon^2 - g_{00} m^2} d\epsilon \approx \frac{2r^2 dr}{\pi [g_{00}]^2} \epsilon^2 d\epsilon. \quad (16)$$

Introducing (16) into (12) we obtain for the Renyi entropy contained in the layer between ρ and $\rho + dr$

$$dH_l = \frac{2\rho^2 dr}{\pi [g_{00}]^2} \int_0^\infty \epsilon^2 d\epsilon W_l(\beta\epsilon) = \frac{2}{\pi \sqrt{g_{00}}} \frac{dr}{\rho} (\rho T_H)^3 \Phi_l = \frac{1}{32\pi^4 [g_{00}]^2} \frac{dr}{\rho} \Phi_l \quad (17)$$

where Φ_l are numerical constants defined as

$$\Phi_l = \int_{\epsilon_0}^\infty z^2 dz W_l(z) ; \quad (18)$$

with W_l given in (13).

We see that the formulae (16) and (17) exhibit a singularity at $r = \rho$ and the procedure to regularize this singularity, described in the Introduction, must be applied. As already mentioned, it turns out that the final result does not depend on the details of regularization.

One can also verify that, when formally integrated from $r = \rho + h$ to ∞ , the divergent part (proportional to $1/h$) is identical to that obtained by 't Hooft [4].

4 Relation to the mass defect

We have evaluated the contribution to the Renyi entropies from the Hawking radiation emitted into an infinitesimal layer of width dr outside of a black hole. At this point the width dr is infinitesimal still arbitrary. To connect it to the physical properties of the black hole we relate it to the change of the black hole mass, dM .

To relate dr to dM we observe that the emission of radiation causes the decrease of the mass of the black hole by the amount of emitted energy reduced by the amount of free energy used in the process of shrinking of the black hole:

$$dM = dE - dF \quad (19)$$

where F is the free energy of the photon gas.

The amount of emitted energy dE can be evaluated from the well-known formula for the Bose gas:

$$dE = dN(\epsilon)\epsilon \frac{e^{-\beta\epsilon}}{1 - e^{-\beta\epsilon}}. \quad (20)$$

Similarly, using the relation between the free energy and the statistical sum Z we have

$$dF = -dN(\epsilon)T \log Z = dN(\epsilon)T \log(1 - e^{-\beta\epsilon}). \quad (21)$$

Using the formula (16) for $dN(\epsilon)$ we obtain

$$dE = \frac{2}{\pi[g_{00}]^2} (\rho T)^3 \frac{dr}{\rho} T \Omega \quad (22)$$

$$dF = \frac{2}{\pi[g_{00}]^2} (\rho T)^3 \frac{dr}{\rho} T \omega \quad (23)$$

where

$$\Omega = \int_{\epsilon_0}^{\infty} z^3 dz \frac{e^{-z}}{1 - e^{-z}}; \quad \omega = \int_{\epsilon_0}^{\infty} z^2 dz \log(1 - e^{-z}). \quad (24)$$

Introducing (22) and (23) into (19) we have

$$dM = \frac{1}{32\pi^4[g_{00}]^2} \frac{dr}{\rho} T(\Omega - \omega) \quad (25)$$

where $g_{00} = g_{00}(r; \rho)$ is defined by (6) and (7).

Consequently,

$$\frac{dr}{\rho} = \frac{32\pi^4[g_{00}]^2}{(\Omega - \omega)} \frac{dM}{T_H}. \quad (26)$$

This formula gives the width of the shell around the horizon where the energy content of the Hawking radiation balances the mass defect of the black hole. To evaluate the Renyi entropies we now use our main idea, assuming that not only energy but also entropy lost by the black hole through radiation is contained in this shell.

5 Renyi entropies of the black hole

When (26) is introduced into (17), one sees that the singular factors $[g_{00}]^2$ in the numerator and denominator cancel exactly and one obtains

$$dH_l = \frac{\Phi_l}{\Omega - \omega} 8\pi M dM \quad (27)$$

where we have used (14). Thus we have expressed the change of the Renyi entropy in terms of the change in the black hole mass, essentially repeating the original procedure of Bekenstein [1, 8].

After integration of (27) from 0 to M we thus have

$$H_l = \frac{\Phi_l}{\Omega - \omega} 4\pi M^2 = \frac{\Phi_l}{\Omega - \omega} S_M, \quad (28)$$

where S_M is the Bekenstein-Hawking entropy of the black hole [1, 2], given by (1). The numerical constants Φ_l, Ω and ω are evaluated in the Appendix and are given by formulae (32), (33), (34) and (35). It is also demonstrated [c.f (36)] that

$$\Phi_1 = \Omega - \omega. \quad (29)$$

With this condition (28) implies the correct formula (1) for the Shannon entropy. This verifies that our conjecture gives the correct estimate of the number of states inside the black hole.

Using now Φ_l given in the Appendix we obtain for the Renyi entropies the formula (8).

6 Conclusions

Our conclusions can be summarized as follows.

(i) We have evaluated the Renyi entropies of a Schwarzschild black hole of radius ρ using the conjecture that they are given by the entropies of the Hawking radiation which fills a thin shell around the horizon. The width of the shell is determined from the condition that the energy of the radiation contained there is equal to the mass defect which the black hole suffers during the emission process. When applied to $H_1 = S$, this procedure reproduces the correct formula (1) for the Bekenstein-Hawking entropy, thus providing the *a posteriori* justification of the argument. The method requires a regularization procedure for the Schwarzschild metric close to the horizon, but the effects of regularization disappear from the final result.

(ii) All Renyi entropies are proportional to the Shannon entropy. The ratio H_l/S turns out to be independent of the mass of the black hole. It is interesting that this ratio is identical to that obtained for the photon gas in a fixed volume.

(iii) Since the Renyi entropies provide additional information about the statistical properties of the black hole, our calculation may perhaps be useful in the search for its internal structure.

(iv) Our calculation is consistent with the idea that the entropy of the black hole is determined by the state of its surface [1, 2, 8]. Indeed, it shows that the black hole entropy can be evaluated from the Hawking radiation at the horizon.

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7 Appendix. Evaluation of numerical constants

To find Φ_l we have to evaluate integral

$$\Theta_l = \int_0^\infty z^2 dz \log(1 - e^{-lz}) \quad (30)$$

This can be done by expansion in series of e^{-lz} :

$$\Theta_l = - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} z^2 dz e^{-nlz} = - \sum_{n=1}^{\infty} \frac{2}{l^3 n^4} = - \frac{\pi^4}{45 l^3} \quad (31)$$

Using this we have

$$\Phi_l = - \frac{l}{l-1} \Theta_1 + \frac{1}{l-1} \Theta_l = \frac{\pi^4}{45} \frac{l^4 - 1}{l^3(l-1)} = \left(1 + \frac{1}{l} + \frac{1}{l^2} + \frac{1}{l^3}\right) \frac{\Phi_1}{4} \quad (32)$$

with

$$\Phi_1 = \frac{4\pi^4}{45}. \quad (33)$$

In a similar way we obtain

$$\Omega = \sum_{n=1}^{\infty} \int_0^{\infty} z^3 dz e^{-nz} = \frac{\pi^4}{15} \quad (34)$$

and

$$\omega = \int_0^{\infty} z^2 dz \log(1 - e^{-z}) = \Theta_1 = - \frac{\pi^4}{45}. \quad (35)$$

Thus, consequently,

$$\Omega - \omega = \Phi_1. \quad (36)$$

References

- [1] J.D.Bekenstein, Phys. Rev. D7 (1973) 2333; Phys. Rev. D9 (1974) 3292.
- [2] S.Hawking, Phys. Rev. D13 (1976) 191; Phys. Rev. D14 (1976) 2460.
- [3] A. Renyi, Proceedings 4th Berkeley Symposium on Mathematical Statistical probability 1960, Vol.1 (University of California Press, Berkeley-Los Angeles, 1961), p. 547.
- [4] G. 't Hooft, Nucl.Phys. B256 (1985) 727.

- [5] A.Bialas and W.Czyz, Acta Phys. Pol. B31 (2000) 2803.
- [6] L.Landau and E.Lifszic, Classical Theory of Fields, Pergamon Press (1975).
- [7] L.Landau and E.Lifszic, Statistical Physics, Pergamon Press (1980).
- [8] For a review see, e.g., L. Susskind, Nature Physics, 2 (2006) 665.